Solution to HW8

Leon Li

Academic Building 1, Room 505

ylli@math.cuhk.edu.hk

§16.4

Circulation and Flux

In Exercises 5–14, use Green's Theorem to find the counterclockwise circulation and outward flux for the field **F** and curve C.

5. $\mathbf{F} = (x - y)\mathbf{i} + (y - x)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

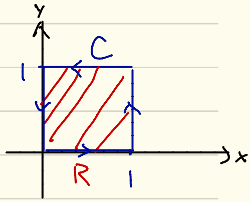
7. $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$

C: The triangle bounded by $y = 0, x = 3$, and $y = x$

14. $\mathbf{F} = \left(\tan^{-1}\frac{y}{x}\right)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j}$

C: The boundary of the region defined by the polar coordinate inequalities $1 \le r \le 2, 0 \le \theta \le \pi$

Sol) (5) $M(x,y) = x - y$; $\frac{\partial M}{\partial x} = 1$; $\frac{\partial M}{\partial y} = -1$



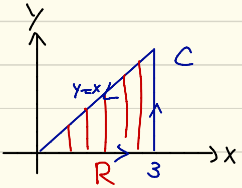
$N(x,y) = y - x$; $\frac{\partial N}{\partial x} = -1$; $\frac{\partial N}{\partial y} = 1$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$; $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2$

$\therefore$ Flux $= \iint_R 2\, dA = \int_0^1 \int_0^1 2\, dx\, dy = 2$

Circulation $= \iint_R 0\, dA = 0$

(7) $M(x,y) = y^2 - x^2$; $\frac{\partial M}{\partial x} = -2x$; $\frac{\partial M}{\partial y} = 2y$



$N(x,y) = x^2 + y^2$; $\frac{\partial N}{\partial x} = 2x$; $\frac{\partial N}{\partial y} = 2y$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2y$; $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = -2x + 2y$

$$\therefore \text{Flux} = \iint_R (-2x+2y)\,dA = \int_0^3 \int_0^x (-2x+2y)\,dy\,dx$$

$$= \int_0^3 \left[-2xy + y^2\right]_0^x dx = \int_0^3 (-x^2)\,dx = \left[\frac{-x^3}{3}\right]_0^3 = -9$$

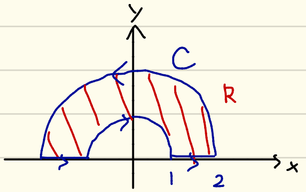
$$\text{Circulation} = \iint_R (2x-2y)\,dA = \int_0^3 \int_0^x (2x-2y)\,dy\,dx$$

$$= -\left(\int_0^3 \int_0^x (-2x+2y)\,dy\,dx\right) = 9$$

(14) $M(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$; $\frac{\partial M}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$; $\frac{\partial M}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$

$N(x,y) = \ln(x^2+y^2)$; $\frac{\partial N}{\partial x} = \frac{1}{x^2+y^2} \cdot 2x = \frac{2x}{x^2+y^2}$; $\frac{\partial N}{\partial y} = \frac{1}{x^2+y^2} \cdot 2y = \frac{2y}{x^2+y^2}$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{x}{x^2+y^2}$; $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \frac{y}{x^2+y^2}$



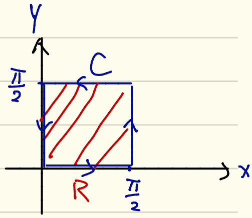
$$\therefore \text{Flux} = \iint_R \frac{y}{x^2+y^2}\,dA = \int_0^{\pi} \int_1^2 \frac{r\sin\theta}{r^2}\,(r\,dr\,d\theta)$$

$$= \left(\int_0^{\pi} \sin\theta\,d\theta\right)\left(\int_1^2 dr\right) = \left[-\cos\theta\right]_0^{\pi} \left[r\right]_1^2 = 2$$

$$\text{Circulation} = \iint_R \frac{x}{x^2+y^2}\,dA = \int_0^{\pi} \int_1^2 \frac{r\cos\theta}{r^2}\,(r\,dr\,d\theta)$$

$$= \left(\int_0^{\pi} \cos\theta\,d\theta\right)\left(\int_1^2 dr\right) = \left[\sin\theta\right]_0^{\pi} \left[r\right]_1^2 = 0$$

16. Find the counterclockwise circulation and the outward flux of the field $\mathbf{F} = (-\sin y)\mathbf{i} + (x\cos y)\mathbf{j}$ around and over the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.



Sol) $M(x,y) = -\sin y$; $\frac{\partial M}{\partial x} = 0$; $\frac{\partial M}{\partial y} = -\cos y$

$N(x,y) = x\cos y$; $\frac{\partial N}{\partial x} = \cos y$; $\frac{\partial N}{\partial y} = -x\sin y$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2\cos y$; $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = -x\sin y$

$\therefore$ Flux $= \iint_R -x\sin y \, dA = -\left(\int_0^{\frac{\pi}{2}} x\,dx\right)\left(\int_0^{\frac{\pi}{2}} (\sin y)\,dy\right) = -\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}}\left[-\cos y\right]_0^{\frac{\pi}{2}} = -\frac{\pi^2}{8}$

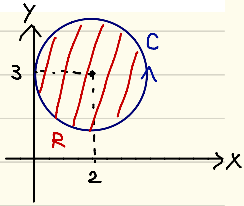
Circulation $= \iint_R 2\cos y \, dA = 2\left(\int_0^{\frac{\pi}{2}} dx\right)\left(\int_0^{\frac{\pi}{2}} \cos y\,dy\right) = 2\left[x\right]_0^{\frac{\pi}{2}}\left[\sin y\right]_0^{\frac{\pi}{2}} = \pi$

Using Green's Theorem

Apply Green's Theorem to evaluate the integrals in Exercises 21–24.

23. $\oint_C (6y + x)\,dx + (y + 2x)\,dy$

C: The circle $(x - 2)^2 + (y - 3)^2 = 4$



Sol) $M(x,y) = 6y + x$; $\frac{\partial M}{\partial x} = 1$; $\frac{\partial M}{\partial y} = 6$

$N(x,y) = y + 2x$; $\frac{\partial N}{\partial x} = 2$; $\frac{\partial N}{\partial y} = 1$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4$

$\therefore$ By Green's Thm, $\oint_C M\,dx + N\,dy = \iint_R (-4)\,dA = -4 \cdot \text{Area}(R) = -16\pi$

Calculating Area with Green's Theorem If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

The reason is that by Equation (3), run backward,

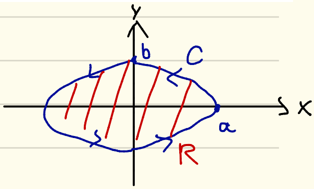
$$\begin{aligned}\text{Area of } R &= \iint_R dy\,dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx \\ &= \oint_C \frac{1}{2}x\,dy - \frac{1}{2}y\,dx.\end{aligned}$$

Use the Green's Theorem area formula given above to find the areas of the regions enclosed by the curves in Exercises 25–28.

26. The ellipse $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

28. One arch of the cycloid $x = t - \sin t, \quad y = 1 - \cos t$

Sol) (26) $x = x(t) = a\cos t$; $dx = -a\sin t\,dt$

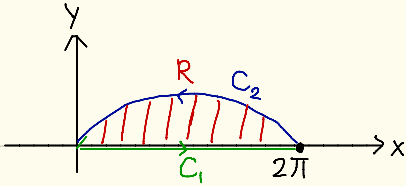


$y = y(t) = b\sin t$; $dy = b\cos t\,dt$

$\therefore$ Area of R $= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t\,dt) - (b\sin t)(-a\sin t\,dt)$

$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$

28) $C_1: x(t) = t,\ t \in [0, 2\pi]\ ;\ dx = dt$



$y(t) \equiv 0 \quad ; \quad dy = 0$

$C_2: x(t) = (2\pi - t) - \sin(2\pi - t) = 2\pi - t + \sin t,\ t \in [0, 2\pi]\ ;\ dx = (-1 + \cos t)\,dt$

$y(t) = 1 - \cos(2\pi - t) = 1 - \cos t,\ \ t \in [0, 2\pi]\ ;\ \ dy = \sin t\, dt$

$\therefore$ Area of $R = \frac{1}{2}\int_{C_1}(x\,dy - y\,dx) + \frac{1}{2}\int_{C_2}(x\,dy + y\,dx)$

$= \frac{1}{2}\left(\int_0^{2\pi}(t)\cdot 0 + 0\ dt)\right) + \frac{1}{2}\int_0^{2\pi}\left((2\pi - t + \sin t)\sin t - (1 - \cos t)(-1 + \cos t)\right)dt$

$= 0 + \frac{1}{2}\int_0^{2\pi}\left((2\pi - t)\sin t + 2 - 2\cos t\right)dt$

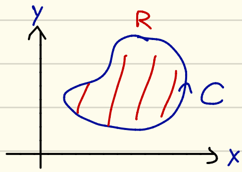
$= \left[-\pi\cos t + t - \sin t\right]_0^{2\pi} + \frac{1}{2}\left(\left[t\cos t\right]_0^{2\pi} - \int_0^{2\pi}\cos t\,dt\right)$

$= (\ 0 + 2\pi - 0\ ) + \frac{1}{2}\left(2\pi - \left[\sin t\right]_0^{2\pi}\right) = 3\pi$

29. Let C be the boundary of a region on which Green's Theorem holds. Use Green's Theorem to calculate

a. $\oint_C f(x)\,dx + g(y)\,dy$

b. $\oint_C ky\,dx + hx\,dy$ (k and h constants).



Sol) (a) $M(x,y) = f(x)$; $\frac{\partial M}{\partial x} = f'(x)$; $\frac{\partial M}{\partial y} = 0$

$N(x,y) = g(y)$; $\frac{\partial N}{\partial x} = 0$; $\frac{\partial N}{\partial y} = g'(y)$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$. $\therefore$ By Green's Thm, $\oint_C (f(x)\,dx + g(y)\,dy) = \iint_R 0\,dA = 0$

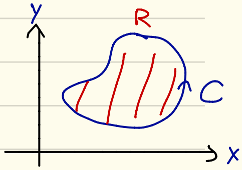
(b) $M(x,y) = ky$; $\frac{\partial M}{\partial x} = 0$; $\frac{\partial M}{\partial y} = k$. $N(x,y) = hx$; $\frac{\partial N}{\partial x} = h$; $\frac{\partial N}{\partial y} = 0$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = h - k$ $\therefore$ By Green's Thm, $\oint_C (ky\,dx + hx\,dy) = \iint_R (h-k)\,dA$

$= (h-k)\cdot \text{Area}(R)$

35. Area and the centroid Let A be the area and $\bar{x}$ the x-coordinate of the centroid of a region R that is bounded by a piecewise smooth, simple closed curve C in the xy-plane. Show that

$$\frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\bar{x}.$$



Sol) Let $\delta(x,y) \equiv 1$. $\bar{x} = \dfrac{\iint_R x\,\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA} = \dfrac{\iint_R x\,dA}{A}$

$\therefore A\bar{x} = \iint_R x\,dA$

① $A\bar{x} = \frac{1}{2}\oint_C x^2 dy$: $A\bar{x} = \iint_R x\,dA = \oint_C \left(0dx + \left(\frac{1}{2}x^2\right) dy\right) = \frac{1}{2}\oint_C x^2 dy$

② $A\bar{x} = -\oint_C xy\,dx$: $A\bar{x} = \iint_R x\,dA = \oint_C (-xy)dx + 0dy = -\oint_C xy\,dx$

③ $A\bar{x} = \frac{1}{3}\oint_C x^2dy - xydx$: $A\bar{x} = \iint_R x\,dA = \oint_C \left(-\frac{xy}{3}\right)dx + \left(\frac{x^2}{3}\right)dy$

$= \frac{1}{3}\left(\oint_C x^2dy - xy\,dx\right)$

$\therefore \frac{1}{2}\oint_C x^2dy = -\oint_C xy\,dx = \frac{1}{2}\oint_C x^2 dy = A\bar{x}$

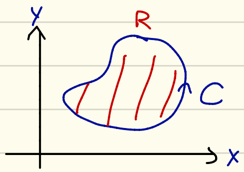
37. Green's Theorem and Laplace's equation Assuming that all the necessary derivatives exist and are continuous, show that if $f(x, y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

then

$$\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0$$

for all closed curves C to which Green's Theorem applies. (The converse is also true: If the line integral is always zero, then f satisfies the Laplace equation.)



Sol) (a) $M(x,y) = \frac{\partial f}{\partial y}$; $\frac{\partial M}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y^2}$

$N(x,y) = -\frac{\partial f}{\partial x}$; $\frac{\partial N}{\partial x} = -\frac{\partial^2 f}{\partial x^2}$; $\frac{\partial N}{\partial y} = -\frac{\partial^2 f}{\partial y \partial x}$

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = 0$

$\therefore$ By Green's Thm, $\oint_C \left(\frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy\right) = \iint_R 0\,dA = 0$